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The High Temperature Ising Model on the Triangular Lattice is a Critical Bernoulli Percolation Model

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Abstract We define a new percolation model by generalising the FK representation of the Ising model, and show that on the triangular lattice and at high temperatures, the critical point in the new model corresponds to the Ising model. Since the new model can be viewed as Bernoulli percolation on a random graph, our result makes an explicit connection between Ising percolation and critical Bernoulli percolation, and gives a new justification of the conjecture that the high temperature Ising model on the triangular lattice is in the same universality class as Bernoulli percolation.

Keywords Ising model · Random-cluster measures · Dependent percolation · DaC models · Sharp phase transition · Duality · $p_c = 1/2$

1 Motivation, Background and Synopsis

If one considers the percolation properties of spin clusters, the high temperature ($\beta < \beta_c$) Ising model on the triangular lattice $\mathbb{T}$ with no external field shows critical behaviour in the sense that the mean spin cluster size is divergent (see, e.g., [3]) and the probability that two spins are in the same spin cluster has a power law decay (see [17]). The relevant universality class is conjectured to be that of two-dimensional Bernoulli (independent) percolation. This conjecture includes convergence of certain interfaces to SLE_6 in the scaling limit, and

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appears in several places; see, e.g., [3, 23, 31, 32]. A renormalisation group analysis of the percolation properties of the Ising model on $\mathbb{T}$, supporting the conjecture, can be found, for instance, in [27] and in Sect. 3 of [29].

In this paper we give a different justification of the above conjecture as follows. The Ising model at inverse temperature β and no external field can be obtained by first drawing a Fortuin-Kasteleyn (FK) random-cluster bond configuration with $q = 2$ and density of open edges $p = 1 - e^{-\beta}$, and then assigning spin $+1$ or -1 to each vertex in such a way that (1) all the vertices in the same FK bond cluster get the same spin and (2) $+1$ and -1 have equal probability. We generalise this procedure by assigning spin $+1$ with probability r and -1 with probability $1 - r$, with $r \in [0, 1]$, while keeping condition (1). For fixed β, we shall view the resulting model as a (dependent) spin percolation model with parameter r, where the only role of the FK model is to introduce dependence between the different spins. We show that, on the triangular lattice and for $\beta < \beta_c$, this model has a percolation phase transition at $r = 1/2$, corresponding to the Ising model.

The model defined above can also be viewed as Bernoulli site percolation on a random graph whose vertices correspond to the FK clusters, and with an edge between two vertices if the corresponding FK clusters are adjacent in $\mathbb{T}$. Therefore, our main result shows that the high temperature Ising model on $\mathbb{T}$ actually *is* critical Bernoulli percolation on a random graph. This interpretation supports the above conjecture and provides a new perspective on the convergence of interfaces to SLE_6 in the scaling limit. As a by-product, we find that for almost every realisation of the above random graph, the percolation critical value is $1/2$.

It is interesting to compare our result to a result of Higuchi, who has extensively studied the percolation properties of the two-dimensional Ising model (see [18–22]). If one considers the Ising model on the square lattice (which we simply denote by $\mathbb{Z}^2$) at inverse temperature β with an external field h, then there exists a critical value $h_c(\beta)$ such that for all $h > h_c(\beta)$, there is an infinite cluster of $+1$ spins, whereas there is no such infinite cluster for $h < h_c(\beta)$. Similarly, one can define $h_c^*(\beta)$ as the critical value for percolation of $+1$ spins in the matching lattice of $\mathbb{Z}^2$ which is obtained by adding diagonal edges inside the faces of $\mathbb{Z}^2$. Higuchi showed in [22] that for all subcritical β, the duality relation $h_c(\beta) + h_c^*(\beta) = 0$ holds. (Note that since it is easier to percolate on the matching lattice than on $\mathbb{Z}^2$, it is not surprising that $h_c^*(\beta)$ is negative.) His proof can be applied to the triangular lattice where, due to the fact that the matching lattice of $\mathbb{T}$ is $\mathbb{T}$ itself, it gives $h_c(\beta) = 0$ for all $\beta < \beta_c$. Thus again the Ising model on $\mathbb{T}$ with no external field corresponds to the critical point of a one-parameter family of models (in this case, the Ising model with an external magnetic field).

Both our result and Higuchi's are meaningful for the conjecture at the beginning of this section in that they relate it to a more general one, namely that all two-dimensional percolation models with "short range" (including exponentially decaying) correlations between sites or bonds belong to the same universality class as Bernoulli percolation (see, e.g., [12]). In particular, it is believed that all such models have the same scaling limit at the critical point. Although this conjecture is widely believed, a general proof seems out of reach at the moment. However, such a strong form of universality has indeed been proved for some specific models of correlated percolation (see [5, 8–11]). For the Ising model with an external field, exponential decay of correlations between sites for all $\beta < \beta_c$ and $h \in \mathbb{R}$ was proved in Theorem 2 in [21], whereas for our model, it immediately follows from the exponential tail decay of the size of FK clusters for $\beta < \beta_c$.

Note, however, that our result is quite different in spirit from Higuchi's. Indeed, a key feature of our model is that the bond clusters are not affected by a change in r, which enables us to view the model as Bernoulli percolation on a random lattice. Such a picture does not hold for the one-parameter family obtained by changing the external field h, since a change in h also affects the FK clusters. In the random lattice interpretation, changing h changes the lattice itself and not just the densities of the two spins. Therefore, our result provides the first direct link between high temperature Ising percolation on $\mathbb{T}$ and critical Bernoulli percolation.

The proofs of the two results are rather different too. They both follow the structure of Russo's formulation [30] of Kesten's proof of $p_c = 1/2$ for independent bond percolation on $\mathbb{Z}^2$, but in both cases, sophisticated methods are required to deal with issues of dependence. Higuchi's proof makes heavy use of the Markovianness of the Ising model for all values of h, but in our model, this property holds only at $r = 1/2$, and fails for all other values of r (see [2]). Therefore we instead utilise the fact that at $r = 1/2$ our model coincides with the Ising model, so that we can use a lemma by Higuchi, together with new domination lemmas (presented in Sect. 4) which enable us to get results for other values of r. This approach relies heavily on the fact that a change in r does not affect the underlying FK clusters that give the correlation structure.

In addition to our main result, we prove uniqueness of the infinite $+1$ cluster for $r > 1/2$, sharpness of the percolation phase transition (by showing exponential decay of the cluster size distribution for $r < 1/2$), and continuity of the percolation function for all $r \in [0, 1]$.

It is a natural and interesting question whether the measure on the spins in our model with parameters β, r and the Ising Gibbs measure with parameters β, h are stochastically ordered (say, for appropriate values of $r > 1/2$ and $h > 0$). A positive answer might provide a different proof of our main result. We (and other people) have thought about this question but were so far unable to give any answer.

We note that van den Berg [34] has recently obtained a general result with an entirely different proof that includes Higuchi's above mentioned result as a special case. However, neither his result nor its proof seem to be sufficient to obtain the results in this paper. Finally, let us mention that percolation questions in a class of models similar to the one studied here are discussed in the recent paper [13].

2 Main Results

We work on the triangular lattice $\mathbb{T}$ with vertex set $\mathcal{V}_{\mathbb{T}}$ and edge set $\mathcal{E}_{\mathbb{T}}$, and denote the unique *Ising Gibbs measure* on $\mathbb{T}$ at inverse temperature $\beta < \beta_c$ and zero external field by μ_β.

The *random-cluster measures* on edge configurations $\eta \in \{0, 1\}^{\mathcal{E}_{\mathbb{T}}}$ (with the usual σ-field generated by cylinder events) are indexed by two parameters satisfying $0 \le p \le 1$ and $q > 0$ (see [16] for the definition and some background). We call an edge $e \in \mathcal{E}_{\mathbb{T}}$ *open* if $\eta(e) = 1$, and *closed* otherwise. The maximal connected components of the graph obtained by removing all the closed edges from $\mathbb{T}$ are called *FK clusters*. For fixed q, there is a percolation phase transition at some $0 < p_c(q) < 1$. If $p < p_c(q)$, with probability one all FK clusters are finite, moreover, there is a unique random-cluster measure which we denote by $\nu_{p,q}$.

When $q = 2$, which we will always assume from now on unless otherwise stated, one can generate an Ising spin configuration $\sigma \in \{+1, -1\}^{\mathcal{V}_{\mathbb{T}}}$ distributed according to μ_β, $\beta < \beta_c$, by drawing an edge configuration according to $\nu_{p,2}$ with $p = 1 - e^{-\beta}$ and assigning spin $+1$ or -1 to each vertex of $\mathbb{T}$ in such a way that (1) all the vertices in the same FK cluster get the same spin and (2) $+1$ and -1 have equal probability. Since $p_c(2) = 1 - e^{-\beta_c}$, $\beta < \beta_c$ implies $p < p_c(2)$, so that $\nu_{p,2}$ is well-defined, and the FK clusters are all finite with probability one.

We generalise the above procedure by assigning spin $+1$ with probability r and -1 with probability $1 - r$, with $r \in [0, 1]$, while keeping condition (1), and denote by $\mathbb{P}_{\beta,r}$ the corresponding measure. For fixed β, this generates a dependent (spin) percolation model with parameter r. Clearly, the spin marginal of $\mathbb{P}_{\beta,1/2}$ coincides with μ_β. Note also that the spin marginal of $\mathbb{P}_{0,1/2}$ (equivalently, μ_0) is a product measure and corresponds to critical site percolation on $\mathbb{T}$. As soon as $\beta > 0$, however, the spins are correlated. Nonetheless, the exponential tail decay of the FK cluster size distribution when $\beta < \beta_c$ (see [16]) immediately implies the exponential decay of correlations in the measure $\mathbb{P}_{\beta,r}$.

We call a maximal connected subset V of $\mathcal{V}_{\mathbb{T}}$ such that all vertices in V have the same spin a *spin cluster*. If the spins in V are all $+1$ (respectively, -1), we call V a $(+)$-*cluster* (resp., a $(-)$-*cluster*). Our aim is to study the percolation properties of spin clusters. We denote by $\Theta(\beta, r)$ the $\mathbb{P}_{\beta,r}$-probability that a given vertex of the triangular lattice is contained in an infinite $(+)$-cluster, and define $r_c(\beta) = \sup\{r : \Theta(\beta, r) = 0\}$. It follows from Proposition 1.8 of [3] that for all $\beta < \beta_c$, we have $r_c(\beta) \geq 1/2$. The main result of this paper is the following theorem.

Theorem 1 *For all $\beta < \beta_c$, $r_c(\beta) = 1/2$.*

Note that, for fixed β, $r = 1/2$ is the "self-dual point" of our model (see e.g. [33] for the meaning and use of self-duality). Thus, Theorem 1 implies that the critical point of the model coincides with its self-dual point. This is in accordance with a very natural principle which is believed to be valid in great generality, but which has been verified only in a handful of cases, including bond percolation on the square lattice [25], site percolation (see [26]) and the Divide and Colour (DaC) model [3] on the triangular lattice, and Voronoi percolation [6]. The same principle should apply to other interesting models, such as the random-cluster model (see [16]), other DaC models (see [3], Conjecture 1.7) and "confetti percolation" (see Problem 5 in [4]).

We point out that, contrary to the model on $\mathbb{T}$ treated in this paper, if one considers the analogous model on the square lattice with a subcritical β, the critical value of r is strictly greater than $1/2$. This follows by standard methods from the exponential tail decay of the size of Ising spin clusters on $\mathbb{Z}^2$ at high temperatures [22, 34].

Theorem 1 concerns the joint measure $\mathbb{P}_{\beta,r}$, but as anticipated, it has implications for the critical value of Bernoulli percolation on the realisations of the random graphs obtained from the FK clusters as explained in Sect. 1. Let E^+ be the event that there exists an infinite $(+)$-cluster somewhere in $\mathbb{T}$. By the ergodicity of $\mathbb{P}_{\beta,r}$ for $\beta < \beta_c$ (which follows from the ergodicity of $\nu_{p,2}$ for all $p < p_c(2)$, see [16]), we have that $\mathbb{P}_{\beta,r}(E^+)$ is 1 for all $r > 1/2$ and 0 for all $r < 1/2$. Now, for a bond configuration $\eta \in \{0, 1\}^{\mathcal{E}_{\mathbb{T}}}$, let G_η denote the realisation of the random graph corresponding to η as defined in Sect. 1. Let $\Theta^{\eta,r}$ denote the probability that in Bernoulli percolation on G_η with density r of $+1$ spins there is an infinite $(+)$-cluster, and define $r_c(\eta) = \sup\{r : \Theta^{\eta,r} = 0\}$. Since $\mathbb{P}_{\beta,r}(E^+) = \int \Theta^{\eta,r} d\nu_{p,2}(\eta)$ with $p = 1 - e^{-\beta}$, the above observation implies that for $\nu_{p,2}$-almost every bond configuration η, the integrand is 0 for all $r < 1/2$ and 1 for all $r > 1/2$. Therefore, we have the following corollary of Theorem 1.

Corollary 1 *For all $\beta < \beta_c$ we have, with $p = 1 - e^{-\beta}$, that*

$$\nu_{p,2}(\{\eta : r_c(\eta) = 1/2\}) = 1.$$

Combining Theorem 1 with results in [3] (see the end of Sect. 5 below), we next obtain the percolation phase diagram in the whole high temperature regime ($\beta < \beta_c$), which is qualitatively the same as for Bernoulli (independent) site percolation on $\mathbb{T}$ (which corresponds to the special case $\beta = 0$ of our model). Due to the $+/-$ symmetry of the model, we focus without loss of generality on the behaviour of $(+)$-clusters. By the size of a cluster we mean the number of vertices in the cluster.

Theorem 2 *For all $\beta < \beta_c$, there is a sharp phase transition at $r_c(\beta) = 1/2$ in the following sense.*

- *If $r < 1/2$, the distribution of the size of the $(+)$-cluster of the origin has an exponentially decaying tail.*
- *If $r = 1/2$, $\Theta(\beta, 1/2) = 0$ and the mean size of the $(+)$-cluster of the origin is infinite.*
- *If $r > 1/2$, there exists a.s. a unique infinite $(+)$-cluster.*

Moreover, for each $\beta < \beta_c$, $\Theta(\beta, r)$ is a continuous function of $r \in [0, 1]$.

A brief outline of the paper is given as follows. In Sect. 3, we introduce some more definitions and notation, and we collect results which are either known or can be proved by standard methods, including a result by Higuchi [21] about the Ising model. We shall use them later, together with the standard Edwards-Sokal coupling [14] and results described in Sect. 4, which contains some technical lemmas and an overview of the proof of Theorem 1. In Sect. 5, we prove Theorem 1 and sketch the proof of Theorem 2.

3 Preliminaries

3.1 Notation and Definitions

In order to define a concrete coordinate system in the triangular lattice $\mathbb{T}$, we embed $\mathbb{T}$ in $\mathbb{R}^2$ as in Fig. 1, so that its set of vertices $\mathcal{V}_{\mathbb{T}}$ consists of the intersections of the lines $y = -\sqrt{3}x + \sqrt{3}k$ and $y = \frac{\sqrt{3}}{2}\ell$ for $k, \ell \in \mathbb{Z}$, and denote the elements of $\mathcal{V}_{\mathbb{T}}$ by (k, ℓ). We call two vertices in $\mathcal{V}_{\mathbb{T}}$ *adjacent* if their Euclidean distance is 1, and define the edge set $\mathcal{E}_{\mathbb{T}}$ by $\mathcal{E}_{\mathbb{T}} = \{(v, w) : v \text{ and } w \text{ are adjacent}\}$.

The state space of our configurations is denoted by $\Omega = \Omega_D \times \Omega_C$, where $\Omega_D = \{0, 1\}^{\mathcal{E}_{\mathbb{T}}}$ is the set of random-cluster realisations, and $\Omega_C = \{-1, +1\}^{\mathcal{V}_{\mathbb{T}}}$ corresponds to the spin

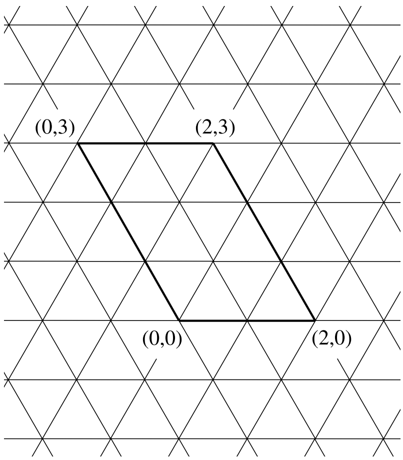

Fig. 1 *Portion of the triangular lattice* $\mathbb{T}$. The heavy segments are the sides of the parallelogram $S_{2,3} = [0, 2] \times [0, 3]$

configurations. The probability measure $\mathbb{P}_{\beta,r}$ is the measure (on the usual σ-algebra on Ω) obtained by the procedure described in Sect. 2; we denote the expectation with respect to $\mathbb{P}_{\beta,r}$ with $\mathbb{E}_{\beta,r}$.

We introduce the set $\tilde{\Omega} \subset \Omega$ as the set of configurations such that vertices in the same FK cluster have the same spin, and we equip $\tilde{\Omega}$ with a relation "$\geq$" (which corresponds to a partial order on Ω_C, hence the notation) as follows. For $\omega_1 = (\eta_1, \sigma_1), \omega_2 = (\eta_2, \sigma_2) \in \tilde{\Omega}$ we say that $\omega_1 \geq \omega_2$ if $\sigma_1(x) \geq \sigma_2(x)$ holds for every $x \in \mathbb{T}$. Note that $\omega_1 \geq \omega_2$ depends on the spins only, and not on the edges. Henceforth all the configurations are implicitly assumed to be in $\tilde{\Omega}$. We call an event $A \subset \tilde{\Omega}$ *increasing* if $\omega \in A$ and $\omega' \geq \omega$ implies $\omega' \in A$. A is a *decreasing event* if A^c is increasing.

We call a sequence $(x_0, x_1, \ldots, x_n)$ of vertices in $\mathbb{T}$ a (*self-avoiding*) *path* if for all $i \in \{0, 1, \ldots, n-1\}$, x_i and x_{i+1} are adjacent, and for all $0 \leq i < j \leq n$, $x_i \neq x_j$. A *horizontal crossing* of a parallelogram $R = [a, b] \times [c, d]$, with $a, b, c, d \in \mathbb{Z}$, is a path $x_0, x_1, \ldots, x_n$ such that $x_0 \in \{a\} \times [c, d]$, $x_n \in \{b\} \times [c, d]$ and for all i, $x_i \in R$. A *vertical crossing* of the same parallelogram is a path $x_0, x_1, \ldots, x_n$ such that $x_0 \in [a, b] \times \{d\}$, $x_n \in [a, b] \times \{c\}$ and for all i, $x_i \in R$.

In a configuration $(\eta, \sigma) \in \tilde{\Omega}$, a $(+)$-*path* is a path $x_0, x_1, \ldots, x_n$ such that for all $i \in \{0, 1, \ldots, n\}$, $\sigma(x_i) = +1$. *Horizontal* $(+)$-*crossings* and *vertical* $(+)$-*crossings* are defined analogously. The definitions of $(-)$-*path*, *horizontal* $(-)$-*crossing*, *vertical* $(-)$-*crossing* are obtained by replacing $+1$ with -1.

Let $S_{n,m}$ denote the parallelogram $[0, n] \times [0, m]$, with $n, m \in \mathbb{N}$. Denote by $V^+_{n,m}$ the event that there is a vertical $(+)$-crossing in $S_{n,m}$; let $H^+_{n,m}$ be the corresponding event with a horizontal $(+)$-crossing. The analogous events with $(-)$-crossings are denoted by $V^-_{n,m}$ and $H^-_{n,m}$, respectively.

Let d denote the graph distance on $\mathbb{T}$. We define the distance between two sets V and W by $d(V, W) = \{\min(d(v, w)) : v \in V, w \in W\}$. Let $B(v, n)$ denote the disc of radius n with center at vertex v in the metric d, i.e., $B(v, n) = \{w : d(v, w) \leq n\}$. For a vertex set $A \subset \mathcal{V}_{\mathbb{T}}$, we denote by ∂A the *internal vertex boundary* of A, that is, we define $\partial A = \{v \in A : \exists w \in \mathcal{V}_{\mathbb{T}} \setminus A$ such that $d(v, w) = 1\}$. For a vertex $v \in \mathcal{V}_{\mathbb{T}}$, let C^{FK}_v be the FK cluster of v, i.e., the set of vertices that can be reached from v through edges that are open in the underlying random-cluster measure with parameters $p = 1 - e^{-\beta}$ and $q = 2$. Let us define the *dependence range* of a vertex v by $\mathcal{D}(v) = \max\{n \in \{0, 1, \ldots\} : C^{FK}_v \cap \partial B(v, n) \neq \emptyset\}$.

We call an edge set $E = \{e_1, e_2, \ldots, e_k\}$ a *barrier* if removing $e_1, e_2, \ldots, e_k$ (but not their end-vertices) separates the graph $\mathbb{T}$ into two or more disjoint connected subgraphs. A barrier E corresponds to one or more dual circuits (obtained by drawing a dual edge perpendicular to e_i through e_i's center for each $e_i \in E$) and its definition is motivated by Lemma 1. Note that exactly one of the resulting subgraphs is infinite, which we call the *exterior* of E, and denote by $\mathrm{ext}(E)$. We call the union of the finite subgraphs the *interior* of E, and denote it by $\mathrm{int}(E)$. With an abuse of notation, we shall write $\mathrm{int}(E)$ and $\mathrm{ext}(E)$ also for the vertex sets of $\mathrm{int}(E)$ and $\mathrm{ext}(E)$ whenever it does not cause confusion. $E = \{e_1, e_2, \ldots, e_k\}$ is a *closed barrier* in a configuration $(\eta, \sigma) \in \tilde{\Omega}$ if E is a barrier and $\eta(e_i) = 0$ holds for all $i \in \{0, 1, \ldots, k\}$. For a vertex set $A \subset \mathcal{V}_{\mathbb{T}}$, let ΔA denote the *edge boundary* of A, that is, $\Delta A = \{(x, y) \in \mathcal{E}_{\mathbb{T}} : x \in A, y \in \mathcal{V}_{\mathbb{T}} \setminus A\}$. Note that for $\beta < \beta_c$, the edge boundary of any FK cluster is a.s. a closed barrier.

3.2 Preliminary Results

To make the paper self-contained, we collect here the tools needed to prove Theorem 1. The first theorem in this subsection follows from results in [1], and is stated explicitly e.g. in [16].

Theorem 3 *If $p < p_c(2)$, there exists $\psi(p) > 0$ such that for all n, we have*

$$\nu_{p,2}(\mathcal{D}(0) \geq n) \leq e^{-n\psi(p)}.$$

Another property of the random-cluster measures is that for $e \in \mathcal{E}_{\mathbb{T}}$ the conditional measure $\nu_{p,q}(\cdot \mid \eta(e) = 0)$ can be interpreted as a random-cluster measure with the same parameters p and q on the graph obtained from $\mathbb{T}$ by deleting e (see [16]). This property implies the following observation, which we state as a lemma for ease of reference.

Lemma 1 *If $B = \{e_1, \dots, e_k\}$ is a barrier, $C(B) = \bigcap_{i=1}^k \{\eta(e_i) = 0\}$, E_1 and E_2 are events which depend only on states of edges and spins of vertices in* $\mathrm{int}(B)$ *and* $\mathrm{ext}(B)$ *respectively, then conditioned on $C(B)$, E_1 and E_2 are independent.*

In the proof of Theorem 1, we will use a version of Russo's formula for decreasing events, hence we state the theorem in a slightly unusual form. The proof, as sketched in [3], is standard. Let A be an event, and let $\omega = (\eta, \sigma)$ be a configuration in $\tilde{\Omega}$. Let C be an FK cluster in η. We call C *pivotal* for the pair (A, ω) if $I_A(\omega) \neq I_A(\omega')$ where I_A is the indicator function of A, $\omega' = (\eta, \sigma')$, and σ' agrees with σ everywhere except that the spins of the vertices in C are different.

Theorem 4 *Let W be a set of vertices with $|W| < \infty$, and let A be a decreasing event that depends only on the spins of vertices in W. Then we have that*

$$\frac{d}{dr}\mathbb{P}_{\beta,r}(A) = -\mathbb{E}_{\beta,r}(n(A)),$$

where $n(A)$ is the number of FK clusters which are pivotal for A.

The following result, like Lemma 2.10 in [3], is a finite size criterion for percolation.

Lemma 2 *There exists a constant $\varepsilon > 0$ with the following property. If β, $p = 1 - e^{-\beta}$ and $N \in \mathbb{N}$ satisfy*

$$(N+1)(3N+1)\nu_{p,2}\left(\mathcal{D}(0) \geq \frac{N}{3}\right) \leq \varepsilon$$

and

$$\mathbb{P}_{\beta,r}(V^{+}_{N,3N}) > 1 - \varepsilon,$$

then $\Theta(\beta, r) > 0$.

As in [3], this lemma can be proved by a coupling argument with a 1-dependent bond percolation model. Theorem 3 and Lemma 2 imply the following result.

Theorem 5 *For all $\beta < \beta_c$, if $\limsup_{n\to\infty} \mathbb{P}_{\beta,r}(V^{+}_{n,3n}) = 1$ for some r, then $\Theta(\beta, r) > 0$.*

3.3 Cut Points

We shall use (a slightly modified version of) a result of Higuchi from [21] (see also Proposition 4.2 in [22]) about the Ising model. In order to state the theorem, we need a few definitions. For positive integer values of k, let $\mathcal{R}_{n,kn}$ be the collection of all horizontal crossings in

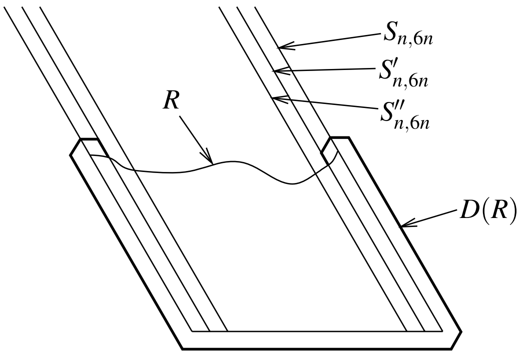

Fig. 2 *Bottom part* of the parallelogram $S_{n,6n}$. The *lines inside* $S_{n,6n}$ parallel to the sides of $S_{n,6n}$ represent the sides of the parallelograms $S'_{n,6n}$ and $S''_{n,6n}$. The *curve from the left side* of $S_{n,6n}$ to its *right side* represents a horizontal crossing $R \in \mathcal{R}_{n,4n}$. The *portion of R contained inside* $S'_{n,6n}$ together with the *thick curve* represents the boundary of $D(R)$

$S_{n,kn}$. For $R \in \mathcal{R}_{n,4n}$, we denote the region in $S_{n,6n}$ (note the different side length) under R by $L(R)$, the region in $S_{n,6n}$ above R by $A(R)$, the parallelogram $[\lfloor n^{1/4} \rfloor, n - \lfloor n^{1/4} \rfloor] \times [0, 6n]$ by $S'_{n,6n}$, and the parallelogram $[2\lfloor n^{1/4} \rfloor, n - 2\lfloor n^{1/4} \rfloor] \times [0, 6n]$ by $S''_{n,6n}$. (For $a \in \mathbb{R}$, we denote by $\lfloor a \rfloor$ the greatest integer smaller than or equal to a.) Also, let $D(R)$ denote the vertex set $\{v \in \mathcal{V}_{\mathbb{T}} \setminus (A(R) \cap S'_{n,6n}) : d(v, L(R) \cup R) \leq n^{1/4}\}$ (see Fig. 2). We call a vertex $x \in R$ a *cut point of R in* $S_{n,6n}$ if there exists a $(+)$-path in $A(R) \cap S''_{n,6n}$ from $[0, n] \times \{6n\}$ to a neighbouring vertex of x (we use Higuchi's language although our definition is slightly different). For a fixed $R \in \mathcal{R}_{n,4n}$, we denote by $c(R)$ the "maximal number of cut points in the middle part of R far enough from each other," that is, the cardinality of a maximal subset $M(R) \subset R \cap S''_{n,6n}$ for which the following properties hold:

– every $v \in M(R)$ is a cut point of R in $S_{n,6n}$,
– for all $v_1, v_2 \in M(R)$, $d(v_1, v_2) \geq \sqrt{n}$.

We shall next compute a lower bound for (a conditional expectation of) $c(R)$ by using the aforementioned result by Higuchi.

Proposition 5.1 in [21] concerning the Ising model on $\mathbb{Z}^2$ essentially states that if both $(+)$-crossings and dual $(-)$-crossings in the long direction of $4n \times n$ rectangles have probability bounded away from 0, then for an arbitrary fixed horizontal crossing R in the lowest quarter of an n by n square S, irrespective of what the spins of vertices in and below R are, the expected number of vertices v in R with a $(+)$-path from a neighbour of v to the top of S is arbitrarily large for all n large enough. A careful reading of the proof of this proposition shows that the same method works on the triangular lattice $\mathbb{T}$. Moreover, we can take the parallelogram $S_{n,6n}$ instead of a square, consider a horizontal crossing $R \in \mathcal{R}_{n,4n}$, condition on the spins of vertices in $D(R)$ instead of $L(R) \cup R$, require that the $(+)$-path from a neighbour of $v \in R$ to the top of $S_{n,6n}$ be in $A(R) \cap S''_{n,6n}$, and still conclude that, under the assumption that $(+)$ and $(-)$-crossings in $S_{3n,n}$ have probabilities bounded away from 0, the expected number of special vertices (which here are cut points of R in $S_{n,6n}$) goes to infinity as $n \to \infty$. In fact, using Higuchi's notation in [21], we see that since all the cut points considered in the proof are found inside annuli A'''_j which are at distance at least $\frac{5}{2} \cdot 4^j$ from one another (where only integers j satisfying $\sqrt{n} \leq 2 \cdot 4^j$ are considered—see (5.21) in [22]), all cut points considered are automatically at distance at least $\frac{5}{4} \cdot \sqrt{n}$ from one another. Therefore, if $\mathbb{E}_\beta$ denotes the expected value w.r.t. μ_β, and $\mathcal{F}_V$ denotes the σ-algebra generated by $\{\sigma(x) : x \in V\}$, we have the following result.

Proposition 1 *Let $\beta < \beta_c$ and assume that there exists $\delta > 0$ such that*

$$\min\{\mu_\beta(H^+_{3n,n}), \mu_\beta(H^-_{3n,n})\} \geq \delta \tag{1}$$

for every $n \geq 1$. Then we have

$$\lim_{n\to\infty} \inf_{R\in\mathcal{R}_{n,4n}} \inf_{E\in\mathcal{F}_{D(R)}} \mathbb{E}_\beta\,(c(R) \mid E) = \infty.$$

Due to the self-matching property of $\mathbb{T}$ and the $+/-$ symmetry of the model, for any $n \in \mathbb{N}$, we have

$$\mu_\beta(H^+_{n,n}) = 1/2. \tag{2}$$

It follows from this observation and the RSW-type results in [18] (which apply to $\mathbb{T}$ as well as to the square lattice) that condition (1) in Proposition 1 is satisfied with a proper choice of δ. Furthermore, since $r = 1/2$ corresponds to the Ising model, for all $\beta < \beta_c$, we have

$$\lim_{n\to\infty} \inf_{R\in\mathcal{R}_{n,4n}} \inf_{E\in\mathcal{F}_{D(R)}} \mathbb{E}_{\beta,1/2}(c(R) \mid E) = \infty. \tag{3}$$

4 Domination Lemmas

4.1 Strategy of the Proof of Theorem 1

In order to motivate the technical results in this section, we give an informal (and somewhat imprecise) overview of proof of $r_c(\beta) \leq 1/2$ for $\beta < \beta_c$. The structure of our proof of this fact is based on Russo's formulation [30] of Kesten's celebrated proof [25] of the analogous statement for bond percolation $\mathbb{Z}^2$. The proof proceeds by contradiction, assuming that $r_c(\beta) > 1/2$ and showing that this implies the existence of some $\varepsilon > 0$ such that, for all $r \in [1/2, 1/2+\varepsilon]$, the number of FK clusters which are pivotal for the event corresponding to the presence of a $(-)$-crossing in a sufficiently large parallelogram $S_{N,6N}$ is very large (in expectation). By Russo's formula, the expected number of pivotal FK clusters equals the derivative of the probability of the crossing event. This leads to a contradiction since the probability of any event has to remain between 0 and 1, and so its derivative cannot be too large on an interval.

We show in Sect. 5 that if we take $\beta < \beta_c$ and assume $r_c(\beta) > 1/2$, then the probability of a horizontal $(-)$-crossing in the lower half $S_{N,3N}$ of the parallelogram $S_{N,6N}$ is bounded away from 0, uniformly for every $r \in [1/2, r_c(\beta))$. We take r_0 in that range and consider the lowest such crossing R and the union U_R of FK clusters of vertices in and below R, which is surrounded by a closed barrier B. Since $\beta < \beta_c$, the FK clusters "tend to be small." Therefore, with high probability, every edge of B is at most at distance $N^{1/4}$ from the set of vertices in and below R. Assuming that this is the case, the internal vertex boundary of $\mathrm{int}(B)$ contains exactly one horizontal crossing of $S_{N,4N}$, which we call Γ_B. Since the vertices in $\Gamma_B \cap S''_{N,6N}$ (i.e., the middle part of Γ_B) are in FK clusters of vertices in the lowest horizontal $(-)$-crossing R, if $v \in \Gamma_B \cap S''_{N,6N}$ is a cut point of Γ_B in $S_{N,6N}$, then C^{FK}_v is pivotal for $H^-_{N,6N}$. Therefore, from this point on, our goal is to find a large number of cut points of Γ_B in $S_{N,6N}$ in $\Gamma_B \cap S''_{N,6N}$.

In Sect. 3.3, we used Higuchi's results and the Edwards-Sokal coupling to obtain (3), which informally states that for $\beta < \beta_c$ and $r = 1/2$, for any horizontal crossing of a sufficiently large parallelogram, regardless of the values of the spins of vertices in and below the crossing, the expected number of cut points of the crossing is arbitrarily large. We would like to use this result to conclude that there are many cut points of Γ_B in $S_{N,6N}$ in $\Gamma_B \cap S''_{N,6N}$. We couple the $r = r_0$ and the $r = 1/2$ case by taking the same random-cluster configuration in $\text{ext}(B)$ (which is allowed since B is a closed barrier), and assigning spins to the FK clusters as follows. We take i.i.d. random variables $(V(C_v^{FK}) : v \in \text{ext}(B))$ with uniform distribution on the interval [0, 1], and assign spin $+1$ to v if $V(C_v^{FK})$ is smaller than r_0 or $1/2$, respectively. Then, every vertex which is a cut point in the $r = 1/2$ case is a cut point in the $r = r_0$ $(> 1/2)$ case as well, since being a cut point requires the presence of $(+)$-paths only, and every vertex in $\text{ext}(B)$ whose spin is $+1$ at $1/2$ has a $+1$ spin also at r_0.

We now would like to use (3), but we cannot do that immediately because at this point of the proof we have information on the FK clusters of vertices in and below R, and not only on spin values, as required by (3). To circumvent this problem, we will use the presence of the closed barrier B to show that having information on the FK clusters of vertices in and below R does not create problems. Proving such a result requires a considerable amount of work, to which the rest of the present section is dedicated.

The proof of $r_c(\beta) \le 1/2$ can be finished from here as follows. First of all, it follows from Lemma 1 that turning the spin of every vertex in $\text{int}(B)$ to -1 does not change the expected number of cut points in $\Gamma_B \cap S''_{N,6N}$. Then, Corollary 2 implies that this expected number is bounded below by the expected number without conditioning on B being closed. For the latter expected number, we can use (3) to conclude that the expected number of cut points in $\Gamma_B \cap S''_{N,6N}$ becomes arbitrarily large as the size of the parallelogram increases, leading to the desired contradiction, as discussed earlier.

4.2 A Barrier Around -1 Spins

Our goal in this section is to prove Corollary 2. We do this through three lemmas, using ideas from [3] and [24]. We need a property of the random-cluster measure $\nu_{p,q}$ on $\mathbb{T}$ from [15] (see also [16]), namely that for all $q \ge 1$, the so-called "FKG lattice condition" holds for $\nu_{p,q}$. We use the following version of it: for any $E \subset \mathcal{E}_{\mathbb{T}}$, $e \in \mathcal{E}_{\mathbb{T}} \setminus E$, and $\psi, \zeta \in \{0,1\}^E$ with $\zeta \ge \psi$ (coordinate-wise), we have

$$\nu_{p,q}(\eta(e) = 1 \mid \eta \equiv \zeta \text{ on } E) \ge \nu_{p,q}(\eta(e) = 1 \mid \eta \equiv \psi \text{ on } E). \tag{4}$$

This property will play an important role in the following proofs. We state the following lemmas for the measure $\mathbb{P}_{\beta,r}$ but in fact all statements in this section hold for all DaC measures obtained by replacing $\nu_{p,2}$ in the construction of $\mathbb{P}_{\beta,r}$ by $\nu_{p,q}$ with $q \ge 1$.

Inequality (4) informally states that the more edges in a certain set E are open, the more likely it is that other edges are open as well. The next lemma states that further conditioning on the left hand side on the event I that the vertices of a certain set V all have the same spin κ leaves the inequality unchanged.

Let $V = \{v_1, v_2, \ldots, v_k\} \subset \mathcal{V}_{\mathbb{T}}$ be a set of vertices, $\kappa \in \{-1, +1\}$ a spin value, $E = \{e_1, e_2, \ldots, e_\ell\} \subset \mathcal{E}_{\mathbb{T}}$ a set of edges, $s_1, s_2, \ldots, s_\ell \in \{0,1\}$ and $g_1, g_2, \ldots, g_\ell \in \{0,1\}$ states, with $g_i \ge s_i$ for all i. Consider the events $I = \bigcap_{i=1}^k \{\sigma(v_i) = \kappa\}$, $A_s = \bigcap_{j=1}^\ell \{\eta(e_j) = s_j\}$, $A_g = \bigcap_{j=1}^\ell \{\eta(e_j) = g_j\}$ (the case $E = \emptyset$, $A_s = A_g = \tilde{\Omega}$ is also allowed).

Lemma 3 *For all $e \in \mathcal{E}_{\mathbb{T}} \setminus E$, we have*

$$\mathbb{P}_{\beta,r}(\eta(e) = 1 \mid A_g, I) \ge \mathbb{P}_{\beta,r}(\eta(e) = 1 \mid A_s). \tag{5}$$

Proof Since

$$\mathbb{P}_{\beta,r}(\eta(e)=1 \mid A_g, I) = \frac{\mathbb{P}_{\beta,r}(\eta(e)=1, A_g, I)}{\mathbb{P}_{\beta,r}(A_g, I)}$$
$$= \frac{\mathbb{P}_{\beta,r}(I \mid \eta(e)=1, A_g)}{\mathbb{P}_{\beta,r}(I \mid A_g)} \cdot \mathbb{P}_{\beta,r}(\eta(e)=1 \mid A_g),$$

and $\mathbb{P}_{\beta,r}(\eta(e)=1 \mid A_g) \geq \mathbb{P}_{\beta,r}(\eta(e)=1 \mid A_s)$ by (4), we have that (5) follows from

$$\mathbb{P}_{\beta,r}(I \mid \eta(e)=1, A_g) \geq \mathbb{P}_{\beta,r}(I \mid A_g). \tag{6}$$

Since

$$\mathbb{P}_{\beta,r}(I \mid A_g) = \mathbb{P}_{\beta,r}(I \mid \eta(e)=1, A_g)\mathbb{P}_{\beta,r}(\eta(e)=1 \mid A_g)$$
$$+ \mathbb{P}_{\beta,r}(I \mid \eta(e)=0, A_g)\mathbb{P}_{\beta,r}(\eta(e)=0 \mid A_g),$$

we see that (6) is equivalent to

$$\mathbb{P}_{\beta,r}(I \mid \eta(e)=1, A_g) \geq \mathbb{P}_{\beta,r}(I \mid \eta(e)=0, A_g). \tag{7}$$

In order to show (7), we will first construct two coupled bond configurations, ψ_0 and ψ_1, such that ψ_0 has distribution $\nu_{p,2}(\cdot \mid \eta(e)=0, A_g)$, ψ_1 has distribution $\nu_{p,2}(\cdot \mid \eta(e)=1, A_g)$ (both with $p = 1-e^{-\beta}$), and $\psi_0 \leq \psi_1$. Such a coupling can be obtained by setting $\psi_0(e_i) = \psi_1(e_i) = g_i$, $\psi_0(e)=0$, $\psi_1(e)=1$, then determining the states of the remaining edges one edge at a time in some deterministic order, using (4) at each step (for a precise way of doing this, see e.g. the proof of Lemma 2 in [24]).

Now notice that given a bond configuration ψ, defining $n(\psi)$ as the number of FK clusters in ψ which contain vertices of V, the probability of I is simply $c^{n(\psi)}$, where $c=r$ if $\kappa=+1$ and $c=1-r$ if $\kappa=-1$. Since $n(\psi_0) \geq n(\psi_1)$ and $0 \leq c \leq 1$, this observation concludes the proof of (7) and thereby the proof of Lemma 3. □

Now take $E = \{e_1, e_2, \ldots, e_\ell\}, s_1, s_2, \ldots, s_\ell, A_s, A_g$ as before Lemma 3, and let $F = \{f_1, f_2, \ldots, f_m\} \subset \mathcal{E}_{\mathbb{T}}$ be a set of edges such that $F \cap E = \emptyset$, and define the event $C(F) = \bigcap_{i=1}^m \{\eta(f_i)=0\}$. Then, as an easy consequence of (4), we have that for all $q \geq 1$, $e \in \mathcal{E}_{\mathbb{T}} \setminus (E \cup F)$,

$$\nu_{p,q}(\eta(e)=1 \mid A_s) \geq \nu_{p,q}(\eta(e)=1 \mid A_s, C(F)).$$

The next lemma follows from this observation and Lemma 3.

Lemma 4 *For all $e \in \mathcal{E}_{\mathbb{T}} \setminus (E \cup F)$, we have*

$$\mathbb{P}_{\beta,r}(\eta(e)=1 \mid A_g, I) \geq \mathbb{P}_{\beta,r}(\eta(e)=1 \mid A_s, C(F)).$$

Note that this statement is still an intuitively clear consequence of (4), since the additional conditioning on I (i.e. that certain vertices all have spin κ) on the left hand side of (4) should intuitively increase the probability that other edges are open, whereas the additional conditioning on $C(F)$ (i.e. having even more edges closed) on the other side should intuitively decrease this probability.

We are now ready to state the main result in this section, which immediately implies the desired Corollary 2. Recall the definition of $\mathcal{F}_V$ for $V \subset \mathcal{V}_{\mathbb{T}}$ right before Proposition 1.

Lemma 5 *Let $V = \{v_1, v_2, \dots, v_k\} \subset \mathcal{V}_{\mathbb{T}}$ be a connected set of vertices, and take its edge boundary $B = \Delta V = \{f_1, f_2, \dots, f_m\} \subset \mathcal{E}_{\mathbb{T}}$ (which is a barrier). Consider the events $I = \bigcap_{i=1}^k \{\sigma(v_i) = -1\}$, $C(B) = \bigcap_{j=1}^m \{\eta(f_j) = 0\}$, and let $D \in \mathcal{F}_{\mathcal{V}_{\mathbb{T}}}$ be an increasing event. Then we have*

$$\mathbb{P}_{\beta,r}(D \mid C(B)) \geq \mathbb{P}_{\beta,r}(D \mid I). \tag{8}$$

Proof We prove (8) by constructing two coupled realisations $(\psi_{C(B)}, \sigma_{C(B)})$ and (ψ_I, σ_I) with distributions $\mathbb{P}_{\beta,r}(\cdot \mid C(B))$ and $\mathbb{P}_{\beta,r}(\cdot \mid I)$ respectively, in such a way that if D occurs in σ_I, it occurs in $\sigma_{C(B)}$ as well.

First, we construct the bond configurations $\psi_{C(B)}$ and ψ_I one edge at a time, using Lemma 4 at each step, as follows. Fix a deterministic order of edges in $\mathcal{E}_{\mathbb{T}}$ starting with edges incident on $v_1, v_2, \dots, v_k$. Take a collection $(U(e) : e \in \mathcal{E}_{\mathbb{T}})$ of i.i.d. random variables having uniform distribution on the interval [0, 1]. We start with a situation where $\psi_{C(B)}(e)$ and $\psi_I(e)$ are undetermined for every edge, and determine the states of edges by the following iteration. We take the first edge in the deterministic order, and denote it by e_1. We declare $\psi_{C(B)}(e_1) = 1$ if and only if $U(e_1) \leq \mathbb{P}_{\beta,r}(\eta(e_1) = 1 \mid C(B))$, and $\psi_I(e_1) = 1$ if and only if $U(e_1) \leq \mathbb{P}_{\beta,r}(\eta(e_1) = 1 \mid I)$. Note that by Lemma 4, $\psi_{C(B)}(e_1) \leq \psi_I(e_1)$.

Let us assume that the states of $e_1, e_2, \dots, e_j$ are determined and $\psi_{C(B)}(e_i) \leq \psi_I(e_i)$ for all $i \in \{1, 2, \dots, j\}$. The next edge e_{j+1} is the next undetermined edge in our deterministic order that shares a vertex with an edge which is open in ψ_I. If no such edge exists, we simply take the next undetermined edge.

Having chosen e_{j+1}, we determine its state by defining $\psi_{C(B)}(e_{j+1}) = 1$ if and only if $U(e_{j+1}) \leq \mathbb{P}_{\beta,r}(\eta(e_{j+1}) = 1 \mid C(B), \bigcap_{i=1}^j \{\eta(e_i) = \psi_{C(B)}(e_i)\})$ (otherwise we assign $\psi_{C(B)}(e_{j+1}) = 0$), and $\psi_I(e_{j+1}) = 1$ if and only if $U(e_{j+1}) \leq \mathbb{P}_{\beta,r}(\eta(e_{j+1}) = 1 \mid I, \bigcap_{i=1}^j \{\eta(e_i) = \psi_I(e_i)\})$ (otherwise $\psi_I(e_{j+1}) = 0$). By the hypothesis $\psi_{C(B)}(e_i) \leq \psi_I(e_i)$ for all $i \in \{1, 2, \dots, j\}$ and Lemma 4, we have that $\psi_{C(B)}(e_{j+1}) \leq \psi_I(e_{j+1})$.

In this way, we obtain bond configurations $\psi_{C(B)}$ with distribution $\mathbb{P}_{\beta,r}(\cdot \mid C(B))$ and ψ_I with distribution $\mathbb{P}_{\beta,r}(\cdot \mid I)$ such that $\psi_{C(B)} \leq \psi_I$. Let us fix j^* to be the index of the last edge chosen by the iteration which is connected by a ψ_I-open edge path to any of the vertices $v_1, v_2, \dots, v_k$. The first part of the iteration (i.e. before e_{j^*+1} is chosen) "explores" the FK clusters in ψ_I of the vertices $v_1, v_2, \dots, v_k$, and when it ends, V is surrounded by a barrier B_2 (which consists of edges from $e_1, e_2, \dots, e_{j^*}$) which is closed in ψ_I. Since $\psi_I \geq \psi_{C(B)}$, B_2 is closed in $\psi_{C(B)}$ as well. Using Lemma 1, we obtain

$$\begin{aligned}
&\mathbb{P}_{\beta,r}\left(\eta(e_{j^*+1}) = 1 \mid C(B), \bigcap_{i=1}^{j^*} \{\eta(e_i) = \psi_{C(B)}(e_i)\}\right) \\
&\quad = \mathbb{P}_{\beta,r}\left(\eta(e_{j^*+1}) = 1 \mid I, \bigcap_{i=1}^{j^*} \{\eta(e_i) = \psi_I(e_i)\}\right),
\end{aligned}$$

which implies $\psi_{C(B)}(e_{j^*+1}) = \psi_I(e_{j^*+1})$. Using the same argument, it is easy to prove by induction that the remaining part of the iteration yields $\psi_{C(B)} = \psi_I$ in $\mathrm{ext}(B_2)$.

We now define the spin configuration σ_I by assigning $+1$ with probability r, -1 with probability $1 - r$ independently to the ψ_I FK clusters in $\mathrm{ext}(B_2)$ (according to some deterministic order), and assigning $\sigma_I(v) = -1$ to each $v \in \mathrm{int}(B_2)$. This gives the correct distribution since every vertex in $\mathrm{int}(B_2)$ is in the same FK cluster as one of the vertices $v_1, v_2, \dots, v_k$. We finish the coupling by defining $\sigma_{C(B)}$ in the following way. We assign

$+1$ with probability r, -1 with probability $1-r$ independently to the $\psi_{C(B)}$ FK clusters in $\mathrm{int}(B_2)$ (according to some deterministic order), and define $\sigma_{C(B)}(v) = \sigma_I(v)$ for all $v \in \mathrm{ext}(B_2)$ (since $\psi_{C(B)} = \psi_I$ in $\mathrm{ext}(B_2)$, we get the right distribution). Let us assume that D occurs in σ_I. It is important to notice that all vertices that have spin $+1$ in σ_I are in $\mathrm{ext}(B_2)$, where $\sigma_{C(B)} = \sigma_I$, so they have spin $+1$ also in $\sigma_{C(B)}$. Since D is an increasing event, this observation shows that D occurs in $\sigma_{C(B)}$ as well. This concludes the proof of Lemma 5. □

Corollary 2 *If $V = \{v_1, v_2, \dots, v_k\} \subset \mathcal{V}_{\mathbb{T}}$ is a connected set of vertices, and $B = \Delta V = \{f_1, f_2, \dots, f_m\} \subset \mathcal{E}_{\mathbb{T}}$ is its edge boundary, then considering the events $I = \bigcap_{i=1}^k \{\sigma(v_i) = -1\}$, $C(B) = \bigcap_{j=1}^m \{\eta(f_j) = 0\}$, and an increasing event $D \in \mathcal{F}_{\mathrm{ext}(B)}$, we have that*

$$\mathbb{P}_{\beta,r}(D \mid C(B), I) \geq \mathbb{P}_{\beta,r}(D \mid I). \tag{9}$$

Proof Since B is a barrier, $I \in \mathcal{F}_{\mathrm{int}(B)}$, and $D \in \mathcal{F}_{\mathrm{ext}(B)}$, we have by Lemma 1 that $\mathbb{P}_{\beta,r}(D \mid C(B), I) = \mathbb{P}_{\beta,r}(D \mid C(B))$. Therefore, Lemma 5 gives the statement. □

5 Proofs of Theorems 1 and 2

By Theorem 5, $r_c(\beta) \leq 1/2$ (and thereby Theorem 1) follows from showing that $\limsup_{n\to\infty} \mathbb{P}_r(V^+_{n,3n}) = 1$ when $r = 1/2 + \varepsilon$ for all $\varepsilon > 0$. We shall prove that the assumption of the contrary implies the presence of too many pivotal FK clusters for a certain event, leading to a contradiction. (For a more detailed summary of the proof, see Sect. 4.1.)

Theorem 6 *For any $\beta < \beta_c$ and $\varepsilon > 0$, we have that*

$$\limsup_{n\to\infty} \mathbb{P}_{\beta,1/2+\varepsilon}(V^+_{n,3n}) = 1.$$

Proof Let us assume that there exist $\beta < \beta_c$, $\varepsilon > 0$ such that

$$\limsup_{n\to\infty} \mathbb{P}_{\beta,1/2+\varepsilon}(V^+_{n,3n}) < 1, \tag{10}$$

and fix such a β and ε. We shall derive a contradiction from (10). Due to the self-matching property of $\mathbb{T}$, (10) implies that there exists $\gamma > 0$ such that for all n large enough,

$$\mathbb{P}_{\beta,1/2+\varepsilon}(H^-_{n,3n}) > \gamma. \tag{11}$$

By (11), monotonicity, (3), and elementary properties of the exponential function, it is possible to choose an integer N large enough so that for $n \geq N$, the following inequalities hold:

$$\mathbb{P}_{\beta,r}(H^-_{n,3n}) > \gamma \quad \forall r \in [1/2, 1/2+\varepsilon], \tag{12}$$

$$\inf_{R\in\mathcal{R}_{n,4n}} \inf_{E\in\mathcal{F}_{D(R)}} \mathbb{E}_{\beta,1/2}(c(R) \mid E) > \frac{2}{\varepsilon\gamma}, \tag{13}$$

$$(n+1)(6n+1)e^{-n^{1/4}\psi(p)} < \frac{\gamma}{2}, \tag{14}$$

where $\psi(p)$ is the same as in Theorem 3. Fix such an N and an arbitrary $r_0 \in [1/2, 1/2+\varepsilon]$. We shall show that, denoting the number of FK clusters which are pivotal for $H^-_{N,6N}$ by $n(H^-_{N,6N})$, we have

$$\mathbb{E}_{\beta,r_0}(n(H^-_{N,6N})) > \frac{1}{\varepsilon}. \tag{15}$$

For $R \in \mathcal{R}_{N,3N}$, we define

$$\begin{aligned} B(R) = \{B \subset \mathcal{E}_{\mathbb{T}} : &B \text{ is a barrier; } \partial(L(R) \cup R) \subset \operatorname{int}(B);\\ &\forall e \in B, d(e, \partial(L(R) \cup R)) \leq N^{1/4};\\ &\partial \operatorname{int}(B) \text{ contains exactly one horizontal crossing of } S_{N,4N}\}. \end{aligned}$$

(The motivation for this definition is that since $\beta < \beta_c$, FK clusters are small, hence with high probability, the "tightest" closed barrier surrounding $L(R) \cup R$ is contained in $B(R)$.) For $B \in B(R)$, we denote the horizontal crossing of $S_{N,4N}$ contained in $\partial \operatorname{int}(B)$ by Γ_B. We also define $l(R)$ to be the event that R is the lowest horizontal $(-)$-crossing in $S_{N,6N}$. For $R \in \mathcal{R}_{N,3N}$, $B \in B(R)$, we denote the union of FK clusters $\bigcup_{v \in L(R) \cup R} C^{FK}_v$ by U_R, the event $\bigcap_{v \in L(R) \cup R} \{\mathcal{D}(v) \leq N^{1/4}\}$ by $t(R)$, and consider the event

$$Q(R, B) = \{l(R)\} \cap \{B = \Delta U_R\} \cap \{t(R)\}.$$

Then we obtain

$$\begin{aligned} \mathbb{E}_{\beta,r_0}(n(H^-_{N,6N})) &\geq \sum_{R \in \mathcal{R}_{N,3N}} \sum_{B \in B(R)} \mathbb{E}_{\beta,r_0}(n(H^-_{N,6N}) \mid Q(R, B)) \mathbb{P}_{\beta,r_0}(Q(R, B))\\ &\geq \sum_{R \in \mathcal{R}_{N,3N}} \sum_{B \in B(R)} \mathbb{E}_{\beta,r_0}(c(\Gamma_B) \mid Q(R, B)) \mathbb{P}_{\beta,r_0}(Q(R, B)) \end{aligned} \tag{16}$$

where the second inequality follows from a pointwise comparison: conditioned on $Q(R, B)$, we have $n(H^-_{N,6N}) \geq c(\Gamma_B)$, due to the following reasons. Using the notation from the definition of $c(\Gamma_B)$ (see Sect. 3.3), conditioned on $Q(R, B)$, the FK cluster of every vertex v in $M(\Gamma_B)$ is pivotal for $H^-_{N,6N}$ since v is a cut point of Γ_B in $S_{N,6N}$, and R is the lowest horizontal $(-)$-crossing in $S_{N,6N}$. It is important to note that every $v \in M(\Gamma_B)$ is indeed in the FK cluster of a vertex in R (i.e., of a vertex in the lowest horizontal $(-)$-crossing), not of a vertex in $L(R)$ (there is no other possibility due to $\{B = \Delta U_R\}$). This is the case since $M(\Gamma_B) \subset \Gamma_B \cap S''_{N,6N}$—since none of the vertices below R has a dependence range larger than $N^{1/4}$, none of the FK clusters of the vertices in $L(R)$ is large enough to go around R and reach the middle part $S''_{N,6N}$ of the parallelogram $S_{N,6N}$. The last step necessary for proving the conditional pointwise comparison is to notice that for $v_1, v_2 \in M(\Gamma_B)$, $v_1 \neq v_2$, we have $C^{FK}_{v_1} \neq C^{FK}_{v_2}$ since $d(v_1, v_2) \geq \sqrt{N}$ and, conditioned on $Q(R, B)$, none of the vertices in $L(R) \cup R$ has a dependence range greater than $N^{1/4}$. Therefore, different vertices in $M(\Gamma_B)$ belong to different pivotal FK clusters.

The next step is to give a lower bound for the expectation via a comparison with the case with parameter $r = 1/2$. We shall first work with probabilities, then we will sum them up to get back the expectation. Let us denote $(N+1)(6N+1)$ (i.e. the number of vertices in $S_{N,6N}$) by K. For a barrier B, we define the events $C(B) = \bigcap_{e \in B} \{\eta(e) = 0\}$ and $W(B) = \bigcap_{v \in \operatorname{int}(B)} \{\sigma(v) = -1\}$. Since for every $R \in \mathcal{R}_{N,3N}$, $B \in B(R)$, $i \in \{1, \ldots, K\}$, we have $\{c(\Gamma_B) \geq i\} \in \mathcal{F}_{A(\Gamma_B) \cap S''_{N,6N}} \subset \mathcal{F}_{\operatorname{ext}(B)}$, $\{l(R) = R\} \in \mathcal{F}_{L(R) \cup R} \subset \mathcal{F}_{\operatorname{int}(B)}$, $W(B) \in \mathcal{F}_{\operatorname{int}(B)}$, and the event $\{B = \Delta U_R\} \cap \{t(R)\}$ depends on the state of edges in $\mathrm{int}(B)$ and B only, it follows from a repeated use of Lemma 1 that for all R, B, and i, we have that

$$\begin{aligned}\mathbb{P}_{\beta,r_0}(c(\Gamma_B) \geq i \mid Q(R, B)) &= \mathbb{P}_{\beta,r_0}(c(\Gamma_B) \geq i \mid C(B)) \\ &= \mathbb{P}_{\beta,r_0}(c(\Gamma_B) \geq i \mid C(B), W(B)).\end{aligned} \tag{17}$$

Coupling the measures with $r = r_0$ and $r = 1/2$ by taking the same bond configurations in $\mathrm{ext}(B)$ (see Sect. 4.1), we see that

$$\mathbb{P}_{\beta,r_0}(c(\Gamma_B) \geq i \mid C(B), W(B)) \geq \mathbb{P}_{\beta,1/2}(c(\Gamma_B) \geq i \mid C(B), W(B)). \tag{18}$$

Since for all i, $\{c(\Gamma_B) \geq i\} \in \mathcal{F}_{\mathrm{ext}(B)}$ is an increasing event, we can use Corollary 2 to conclude that

$$\mathbb{P}_{\beta,1/2}(c(\Gamma_B) \geq i \mid C(B), W(B)) \geq \mathbb{P}_{\beta,1/2}(c(\Gamma_B) \geq i \mid W(B)). \tag{19}$$

Summing up for $i \in \{1, 2, \ldots, K\}$, using (17), (18), (19) and then (13), we obtain that for every $R \in \mathcal{R}_{N,3N}$, $B \in B(R)$, a.s.,

$$\begin{aligned}\mathbb{E}_{\beta,r_0}(c(\Gamma_B) \mid Q(R, B)) &= \sum_{i=1}^{K} \mathbb{P}_{\beta,r_0}(c(\Gamma_B) \geq i \mid Q(R, B)) \geq \sum_{i=1}^{K} \mathbb{P}_{\beta,1/2}(c(\Gamma_B) \geq i \mid W(B)) \\ &= \mathbb{E}_{\beta,1/2}(c(\Gamma_B) \mid W(B)) > \frac{2}{\varepsilon\gamma}.\end{aligned} \tag{20}$$

Finally we need to note that for a crossing $R \in \mathcal{R}_{N,3N}$, if $t(R)$ occurs, then $\Delta U_R \in B(R)$. Therefore,

$$\begin{aligned}&\sum_{R\in\mathcal{R}_{N,3N}} \sum_{B\in B(R)} \mathbb{P}_{\beta,r_0}(Q(R, B)) \\ &\quad= \sum_{R\in\mathcal{R}_{N,3N}} \mathbb{P}_{\beta,r_0}(l(R) \cap t(R)) \geq \mathbb{P}_{\beta,r_0}(H^-_{N,3N}) - \mathbb{P}_{\beta,r_0}\left(\bigcup_{v\in R_{n,6n}} \mathcal{D}(v) > N^{1/4}\right) \\ &\quad\geq \gamma - (N + 1)(6N + 1)\nu_{p,2}(\mathcal{D}(0) > N^{1/4}) \geq \gamma - (N + 1)(6N + 1)e^{-N^{1/4}\psi(p)} \geq \gamma/2,\end{aligned}$$

where we used the translation invariance of $\nu_{p,2}$, (12), Theorem 3, and (14). Using (16), (20), and this computation, we obtain that

$$\mathbb{E}_{\beta,r_0}(n(H^-_{N,6N})) > \sum_{R\in\mathcal{R}_{N,3N}} \sum_{B\in B(R)} \frac{2}{\varepsilon\gamma} \mathbb{P}_{\beta,r_0}(Q(R, B)) \geq \frac{2}{\varepsilon\gamma} \cdot \frac{\gamma}{2} = \frac{1}{\varepsilon},$$

as desired.

Since (15) can be proved for all $r \in [1/2, 1/2 + \varepsilon]$ with the same method, we obtain by Theorem 4 that

$$\sup_{r\in[1/2,1/2+\varepsilon]} \frac{d}{dr} \mathbb{P}_{\beta,r}(H^-_{N,3N}) < -\frac{1}{\varepsilon},$$

which leads to a contradiction since it yields

$$\mathbb{P}_{\beta,1/2+\varepsilon}(H^-_{N,3N}) \le \mathbb{P}_{\beta,1/2}(H^-_{N,3N}) + \varepsilon \sup_{r\in[1/2,1/2+\varepsilon]} \frac{d}{dr}\mathbb{P}_{\beta,r}(H^-_{N,3N})$$
$$< \mathbb{P}_{\beta,1/2}(H^-_{N,3N}) - 1. \qquad \square$$

Sketch of the proof of Theorem 2. The exponential tail of the distribution of the size of the $(+)$-cluster of the origin for $r < 1/2$ can be proved similarly to Theorem 2 in [6]. The statement concerning the critical case $r = 1/2$ has been proved in Proposition 1.8 of [3]. We mention that one can obtain a polynomial lower bound for the tail distribution of the $(+)$-cluster of the origin at $r = 1/2$ by using elementary duality arguments only, see [17, p. 15]. The ergodicity of $\mathbb{P}_{\beta,r}$ for $\beta < \beta_c$ guarantees the presence of an infinite $(+)$-cluster when $r > 1/2$.

The uniqueness of the infinite $(+)$-cluster follows from a result in [7], which implies that if a probability measure μ on $\{-1,+1\}^{\mathcal{V}_\mathbb{T}}$ is translation invariant and satisfies the finite energy condition [28], then μ-a.s. there exists at most one infinite cluster of $+1$'s. If $\beta < \infty$ and $0 < r < 1$, then the spin marginal of $\mathbb{P}_{\beta,r}$ clearly satisfies both properties.

The statement about the continuity of $\Theta(\beta,r)$ in r for $\beta < \beta_c$ follows from $\Theta(\beta,1/2) = 0$ and the uniqueness of the infinite $(+)$-cluster by standard methods (see [35]), in the same way as the analogous result in [3].

Remark 1 In all the proofs in this paper, the FKG inequality and RSW-type arguments are used for $\mathbb{P}_{\beta,r}$ only *at* the critical point $r = 1/2$, never away from it. This way of proving classical percolation results can be useful in the case of models, like the present one, where the (conjectured) critical point has special properties and is better understood compared to other values of the parameter.

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References

1. Aizenman, M., Barsky, D.J., Fernández, R.: The phase transition in a general class of Ising-type models is sharp. J. Stat. Phys. **47**, 343–374 (1987)
2. Bálint, A.: Gibbsianness and non-Gibbsianness in divide and colour models. Ann. Probab. (2009, to appear)
3. Bálint, A., Camia, F., Meester, R.: Sharp phase transition and critical behaviour in 2D divide and colour models. Stoch. Process. Appl. **119**, 937–965 (2009)
4. Benjamini, I., Schramm, O.: Exceptional planes of percolation. Probab. Theory Relat. Fields **111**, 551–564 (1998)
5. Binder, I., Chayes, L., Lei, H.K.: Conformal invariance for certain models of the bond-triangular type. Available at arXiv:0710.3446v3 [math-ph] (2009)
6. Bollobás, B., Riordan, O.: The critical probability for random Voronoi percolation in the plane is 1/2. Probab. Theory Relat. Fields **136**, 417–468 (2006)

7. Burton, R., Keane, M.: Density and uniqueness in percolation. Commun. Math. Phys. **121**, 501–505 (1989)
8. Camia, F.: Scaling limit and critical exponents for 2D bootstrap percolation. J. Stat. Phys. **118**, 85–101 (2005)
9. Camia, F.: Universality in two-dimensional enhancement percolation. Random Struct. Algorithms **33**, 377–408 (2008)
10. Camia, F., Newman, C.M.: The percolation transition in the zero-temperature Domany model. J. Stat. Phys. **114**, 1199–1210 (2004)
11. Camia, F., Newman, C.M., Sidoravicius, V.: A particular bit of universality: scaling limits for some dependent percolation models. Commun. Math. Phys. **246**, 311–332 (2004)
12. Cardy, J.: Lectures on conformal invariance and percolation. Available at arXiv:math-ph/0103018 (2001)
13. Chayes, L., Lebowitz, J.L., Marinov, V.: Percolation phenomena in low and high density systems. J. Stat. Phys. **129**, 567–585 (2007)
14. Edwards, R.G., Sokal, A.D.: Generalization of the Fortuin-Kasteleyn-Swendsen-Wang representation and Monte Carlo algorithm. Phys. Rev. D **38**, 2009–2012 (1988)
15. Fortuin, C.M., Kasteleyn, P.W., Ginibre, J.: Correlation inequalities on some partially ordered sets. Commun. Math. Phys. **22**, 89–103 (1971)
16. Grimmett, G.: The Random-Cluster Model. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 333. Springer, Berlin (2006)
17. Grimmett, G., Janson, S.: Random even graphs. Electron. J. Comb. **16**, R46 (2009)
18. Higuchi, Y.: A weak version of RSW theorem for the two-dimensional Ising model. Contemp. Math. **41**, 207–214 (1985)
19. Higuchi, Y.: Percolation of the two-dimensional Ising model. In: Stochastic Processes-Mathematics and Physics, II (Bielefeld, 1985). Lecture Notes in Math., vol. 1250, pp. 120–127. Springer, Berlin (1987)
20. Higuchi, Y.: A remark on the percolation for the 2D Ising model. Osaka J. Math. **26**, 207–224 (1989)
21. Higuchi, Y.: Coexistence of infinite ($*$)-clusters. II. Ising percolation in two dimensions. Probab. Theory Relat. Fields **97**, 1–33 (1993)
22. Higuchi, Y.: A sharp transition for the two-dimensional Ising percolation. Probab. Theory Relat. Fields **97**, 489–514 (1993)
23. Kager, W., Nienhius, B.: A guide to stochastic Löwner evolution and its applications. J. Stat. Phys. **115**, 1149–1229 (2004)
24. Kahn, J., Weininger, N.: Positive association in the fractional fuzzy Potts model. Ann. Probab. **35**, 2038–2043 (2007)
25. Kesten, H.: The critical probability of bond percolation on the square lattice equals $1/2$. Commun. Math. Phys. **74**, 41–59 (1980)
26. Kesten, H.: Percolation Theory for Mathematicians. Birkhäuser, Boston (1982)
27. Klein, W., Stanley, H.E., Reynolds, P.J., Coniglio, A.: Renormalization-group approach to the percolation properties of the triangular Ising model. Phys. Rev. Lett. **41**, 1145–1148 (1978)
28. Newman, C.M., Schulman, L.S.: Infinite clusters in percolation models. J. Stat. Phys. **26**, 613–628 (1981)
29. Qian, X., Deng, Y., Blöte, H.W.J.: Percolation in one of q colors near criticality. Phys. Rev. B **71**, 144303 (2005)
30. Russo, L.: On the critical percolation probabilities. Z. Wahrscheinlichkeitstheor. Verw. Geb. **56**, 229–237 (1981)
31. Sheffield, S.: Exploration trees and conformal loop ensembles. Duke Math. J. **147**, 79–129 (2009)
32. Smirnov, S.: Towards Conformal Invariance of 2D Lattice Models. International Congress of Mathematicians, vol. II, pp. 1421–1451. Eur. Math. Soc., Zürich (2006)
33. Sykes, M.F., Essam, J.W.: Exact critical percolation probabilities for site and bond problems in two dimensions. J. Math. Phys. **5**, 1117–1127 (1964)
34. van den Berg, J.: Approximate zero-one laws and sharpness of the percolation transition in a class of models including 2D Ising percolation. Ann. Probab. **36**, 1880–1903 (2008)
35. van den Berg, J., Keane, M.: On the continuity of the percolation probability function. Contemp. Math. **26**, 61–65 (1984)